

STABILITY OF LU-KUMAR NETWORKS UNDER LONGEST-QUEUE AND LONGEST-DOMINATING-QUEUE SCHEDULING

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Abstract

We consider the stability of robust scheduling policies for Lu-Kumar networks. These are open networks with arbitrary routing matrix and several disjoint groups of queues in which at most one queue can be served at a time in each group. The arrival and potential service processes and routing decisions at the queues are independent, stationary and ergodic. A scheduling policy is called *robust* if it does not depend on the arrival and service rates nor on the routing probabilities. A policy is called *throughput-optimal* if it makes the system stable whenever the parameters are such that the system can be stable. We propose two robust policies: longest-queue scheduling and a new policy called longest-dominating-queue scheduling. We show that longest-queue scheduling is throughput-optimal for two groups of two queues. We also prove the throughput optimality of longest-dominating-queue scheduling when the network topology is acyclic, for an arbitrary number of groups and queues.

Keywords: Stability; Longest-queue scheduling; Queueing networks; Fluid limit; Lu-Kumar network

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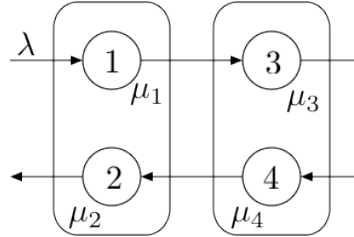


FIGURE 1: Example of Lu-Kumar network

1. Introduction

We consider the scheduling of Lu-Kumar networks [1], [9]. These are queueing networks with disjoint groups of queues that cannot be scheduled simultaneously. MaxWeight scheduling, proposed in [3], is known to be throughput optimal for these networks. However, MaxWeight scheduling suffers from high complexity and also dependency on the knowledge of all of the service rates, queue lengths, and routing probabilities. A natural low complexity scheduling algorithm is longest-queue (LQ) scheduling, studied in [2], [7] and [11], which we discuss in this paper in details.

In general, we know that the utilization being less than one for each server is necessary but not sufficient for the stability of a queueing network. This condition specifies that work arrives at each server at rate less than one. In [1], Lu and Kumar provided an example of a network (Figure 1) with priority scheduling that is unstable despite satisfying the utilization condition. To see this, assume that $\mu_1 > \mu_3$ and $\mu_4 > \mu_2$ and that queue 3 is initially empty while queue 2 is not. Group 1 serves queue 2, so that queue 1 is not served and queue 3 remains empty. Eventually, queue 2 becomes empty and queues 1 and 3 get served until queue 3 becomes empty (because $\mu_1 > \mu_3$). At that time, queues 4 and 2 get served, until queue 2 becomes empty (because $\mu_4 > \mu_2$). Thus, queues 2 and 3 are never served together. Consequently, they form a virtual group and the system cannot be stable unless the utilization of that virtual group is less than one, which is an additional condition not implied by the original utilization condition.

If priority is given to the last buffer in each station (queues 2 and 4), the network is stable. It is shown in [5, 9] that, for a re-entrant line, the last-buffer-first-serve discipline with pre-emption is throughput optimal.

We analyze the stability of the queueing networks under robust policies using their fluid model. The fluid model converts a stochastic system into a deterministic system, based on the functional strong law of large numbers, [4, 6, 8, 9, 12]. Under weak assumptions, the stability of the fluid model implies the stability of the queueing network. The common method for showing stability of the fluid model is to find a Lyapunov function for the differential equations of the system. A novel feature of this paper is that to prove the stability of the fluid system under LQ scheduling, we investigate the possible trajectories of the state and show that they can only go to zero, instead of using a direct Lyapunov function.

There has been relatively little work on the stability of LQ scheduling in the literature. We can mainly mention the following papers. In [10], Kumar et al. consider sufficient conditions for stability of stable marriage scheduling algorithms in input-queued switches. In [11], Dimakis and Walrand identify new sufficient conditions for longest-queue-first (LQF) scheduling to be throughput optimal. They combine properties of diffusion-scaled path functionals and local fluid limits into a sharper characterization of stability. See also [13] for a variation on the first order sufficient condition of that paper (resource pooling). Recently, in [14], Baharian and Tezcan consider LQF scheduling for parallel server systems. It is shown that the nominal traffic condition is sufficient to prove stability if the underlying graph of the parallel server system is a tree. Furthermore, additional drift conditions are provided for the stability of a special parallel server system known as X-model. The network model that we consider is different from all the previous work on LQ scheduling.

In addition to LQ scheduling, we propose a new scheduling policy called longest-dominating-queue (LDQ). This policy is closely related to LQ scheduling. According to this policy, none of the queues that feed a larger maximum-length queue of another group is served. Among those queues that are not dominated by a maximum-length queue of another group, the longest one is scheduled. We use the fluid model and show that the maximum of the queue lengths is a Lyapunov function to prove the stability of LDQ scheduling in a general network which is an acyclic graph.

We use boldface lower case letters \mathbf{v} to denote vectors and boldface capital letters \mathbf{M} to denote matrices. $[\cdot]^T$ denotes matrix transposition operation and $[\cdot]^{-T}$ the matrix inversion and transposition. Let \mathbf{V} and \mathbf{W} be matrices of size $L \times N$, then $\mathbf{V} \preceq \mathbf{W}$

means $v_{ij} \leq w_{ij}$, for $i = 1, \dots, L$ and $j = 1, \dots, N$. Also, $\mathbf{1}_L$ and $\mathbf{0}_L$ stand for column vectors of 1's and 0's with length L , respectively and $\mathbf{0}_{L \times N}$ stands for the $L \times N$ matrix of all 0 entries. The indicator function of set A is shown as 1_A .

The rest of the paper is organized as follows. In Section 2, we provide the precise network model and problem formulation, and state the main results of the paper. In Section 3, we focus on the LQ scheduling and prove that it is throughput optimal for Lu-Kumar networks with two groups of two queues. We also provide sufficient conditions for the stability of LQ scheduling for Lu-Kumar networks. In Section 4, we propose the LDQ scheduling and prove that this new policy is stable if the network topology is acyclic. We provide simulation results and examples which show that when there is a cycle in the network, LDQ may not be stable.

2. Problem definition and system model

2.1. Network model

We consider a network with K queues and a routing matrix $\mathbf{R}_{K \times K}$. The entry r_{ij} is the probability that a job goes to queue j upon leaving queue i . Therefore, a job leaves the network upon leaving queue i with probability $1 - \sum_j r_{ij}$. In each queue, jobs are served in their order of arrival.

The random length of queue i at time $t \geq 0$ is $\bar{X}_i(t)$, the vector of service rates of the queues is $\boldsymbol{\mu} = [\mu_1, \mu_2, \dots, \mu_K]^T$, and the vector of arrival rates to the queues is $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \dots, \lambda_K]^T$. The arrival processes to the queues are independent stationary ergodic processes. In this network, not all the queues can be served at the same time. The queues are partitioned into J disjoint groups $\{\mathcal{G}_j\}_{j=1}^J$ and only one queue can be served in each group at a time. Let $K_j = |\mathcal{G}_j|$, so that $K_1 + K_2 + \dots + K_J = K$. We call such networks *Lu-Kumar networks* because they were analyzed by Lu and Kumar in [1]. Figure 2 illustrates such a network with 2 groups of two queues: $\mathcal{G}_1 = \{1, 2\}$ and $\mathcal{G}_2 = \{3, 4\}$.

We assume the network is open, i.e., all the jobs eventually leave the network. Since the network is open,

$$\mathbf{Q} = \mathbf{I} + \mathbf{R}^T + (\mathbf{R}^T)^2 + \dots = (\mathbf{I} - \mathbf{R}^T)^{-1}$$

is a finite positive matrix. So, the matrix $(\mathbf{I} - \mathbf{R}^T)$ is invertible.

The service disciplines that we consider are nonpreemptive and independent of λ , μ and \mathbf{R} . The goal is to analyze the stability of this network when the traffic conditions stated in Section 2.2 hold. We use the fluid model to analyze the stability of the network. The fluid model is described by a set of ordinary differential equations. Let $\bar{X}_i(t)$ be the length of queue i at time t and $\bar{T}_i(t)$ be the random total amount of time that queue i has been scheduled up to time t . Let also $X_i(t) = \lim_{r \rightarrow \infty} \bar{X}_i(rt)/r$ and $T_i(t) = \lim_{r \rightarrow \infty} \bar{T}_i(rt)/r$. Then the fluid model equations are

$$X_j(t) = X_j(0) + \lambda_j t - \mu_j T_j(t) + \sum_{i=1}^K r_{ij} \mu_i T_i(t), \quad j = 1, \dots, K. \quad (1)$$

The fluid model is *stable* if there exists some $\delta > 0$ such that, for each fluid solution with $|\mathbf{X}(0)| \leq 1$, one has $|\mathbf{X}(t)| = 0$ for $t > \delta$. Under weak conditions, the stability of the fluid model implies the stability of the queueing network (e.g., the Harris recurrence of a Markov model in the case of renewal arrival processes, i.i.d. service times and Markov routing). See [9] for a discussion of such results.

To investigate the possible trajectories of the differential equations, we define the *state* $S(t)$ of the network at time $t \geq 0$ to be the set of queues with maximum queue length in each group at that time. More precisely, if S_j is the set of maxima in group j , for $j = 1, \dots, J$, the state of the network is $S = (S_1, S_2, \dots, S_J)$. Since S_j is a subset of $\mathcal{G}_j = \{1, 2, \dots, K_j\}$, there are $\prod_{j=1}^J 2^{K_j} = 2^K$ possible states of the network. Note that the state of a group is the empty set if all the queues in that group are empty. A state (S_1, S_2, \dots, S_J) is said to be *feasible* if $S(t)$ can spend a positive amount of time in that state. Later we see that following the trajectory of the state $S(t)$ is the key idea to prove the stability of LQ scheduling in Lu-Kumar networks.

The main contributions of this paper are the following theorems.

Theorem 2.1. *The Lu-Kumar network in Figure 2, with two groups of two queues, is stable under LQ scheduling if the utilization conditions hold.*

Theorem 2.2. *A Lu-Kumar networks is stable under LDQ scheduling if the utilization conditions hold and there is no cycle in the topology of the network.*

The stability of Lu-Kumar networks with more than two groups of two queues under

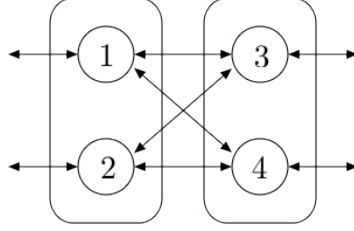


FIGURE 2: Lu-Kumar network with two groups of two queues

LQ scheduling is still an open question. Examples show that a Lu-Kumar network may not be stable under LDQ scheduling if its topology is not acyclic.

2.2. Stability condition

Define the drift matrix \mathbf{D} of the network by

$$\mathbf{D} = \mathbf{M}(\mathbf{R} - \mathbf{I}), \quad (2)$$

where $\mathbf{M} = \text{diag}\{\boldsymbol{\mu}\}$. Since $\mathbf{Q} := (\mathbf{I} - \mathbf{R}^T)^{-1}$ is a positive matrix, one has

$$\mathbf{D}^{-T} = -\mathbf{M}^{-1}\mathbf{Q} \preceq \mathbf{0}_{K \times K}. \quad (3)$$

Furthermore, let the vector of nominal traffic of the queues be $\boldsymbol{\nu} = [\nu_1, \nu_2, \dots, \nu_K]^T$.

We have

$$\nu_i = \lambda_i + \sum_{j=1}^K r_{ji} \nu_j.$$

In matrix form,

$$\boldsymbol{\nu} = (\mathbf{I} - \mathbf{R}^T)^{-1} \boldsymbol{\lambda}. \quad (4)$$

By Proposition 2.5.3. in [9], the necessary stability conditions of the network are

$$\sum_{i \in \mathcal{G}_j} \frac{\nu_i}{\mu_i} < 1, \quad \forall j = 1, \dots, J. \quad (5)$$

These are the utilization conditions. They express that work for each group arrives at the network at rate less than one.

Using Equations (2) and (4), we find an equivalent stability conditions which is that $\tilde{\mathbf{e}}_j^T \mathbf{D}^{-T} \boldsymbol{\lambda} + 1 > 0$ for all $j = 1, \dots, J$, where $\tilde{\mathbf{e}}_j$ is a column vector of length K corresponding to group j such that component i of $\tilde{\mathbf{e}}_j$ is $\tilde{e}_j(i) = 1_{\{i \in \mathcal{G}_j\}}$.

3. LQ scheduling

3.1. General network

In this section, we prove a lemma that will be useful in the proofs for the stability of LQ and LDQ scheduling. We also provide sufficient conditions for the stability of general networks under LQ scheduling. Under this discipline, the longest queue in each group is served. To break ties, we can use a static priority scheduling among the maximum queues, or serve any of the maximum-length queues randomly.

We use the fluid model to prove the stability of LQ scheduling as stated in Equation (1). Further conditions for LQ scheduling are that the server in each group spends all its time serving the longest queues. That is, if S_j is the set of non-empty queues with maximum length in group \mathcal{G}_j , then

$$\begin{aligned} \sum_{i \in S_j} \dot{T}_i(t) &= 1 \\ \sum_{i \in \mathcal{G}_j \setminus S_j} \dot{T}_i(t) &= 0. \end{aligned} \tag{6}$$

A regular point is a value of $t \in [0, \infty)$ at which the function

$$f : [0, \infty) \rightarrow \mathbb{R}^J, \quad f(t) = [\max_i \{X_i\}_{i \in \mathcal{G}_1}, \dots, \max_i \{X_i\}_{i \in \mathcal{G}_J}]^T$$

is differentiable in all of its components. Therefore, in a regular point S_t remains constant in $[t, t + \epsilon]$ for some $\epsilon > 0$. By S_t we indicate the value of S at time $t \geq 0$.

Let S_j be the set of queues with maximum length in group j and $|S_j| = L_j$. Define $S = \cup_j S_j$ and $|S| = L = \sum_j L_j$. The corresponding drift matrix and arrival vectors to set S are $\mathbf{D}_L = [d_{i,k}]_{i,k \in S}$ and $\boldsymbol{\lambda}_L = [\lambda_i]_{i \in S}$. Also, let $\dot{f}(t) = \boldsymbol{\alpha} = [\alpha_1, \dots, \alpha_J]^T$ be the drifts of maximum length queues in the J groups.

A sketch of the proof for stability of LQ scheduling in Lu-Kumar networks with two groups of two queues is the following. The drifts of queues with maximum length in one group are equal at a regular point by Lemma 3.1. In Lemma 3.3, we show that $\dot{f}(t)$ is strictly negative in at least one of its components, at a regular point. More precisely, if t is a regular point, $\dot{f}(t)$ has a component i determined by S_t such that $\dot{f}_i(t) \leq -\epsilon(S_t)$ for some $\epsilon(S_t) > 0$ if $|\mathbf{X}(t)| > 0$. So S_t cannot remain constant as $t \rightarrow \infty$ unless $\mathbf{X}(t) = \mathbf{0}$, but this is not enough to prove stability, as S_t can in principle travel

around different states forever. We will show that there is no loop in the trajectory of S_t , and this will establish the result.

Note that S_t goes from one state to another when a new queue appears in the set of maxima in one group. Indeed, state S is feasible if time-sharing can keep the queue-lengths of set of maxima equal in that state, so that a queue cannot disappear from the set of maxima. Later we see that S is feasible if its corresponding vector of time-sharing $\dot{\mathbf{T}}$ obtained from Equation (9) below is a non-negative vector.

We define two possible moves for S_t in the state diagram of the network.

1. *Increase*: A new maximum appears in one of the groups and S_t reaches a feasible state.
2. *Jump*: A new maximum appears in one of the groups and S_t visits an infeasible state but immediately goes to another feasible state.

To clarify the two possible moves, we consider an example of the network in Figure 1. Suppose that S_t is initially in state $(1, 4)$ and that, after some time, X_2 becomes equal to X_1 . At this time, queues 1 and 2 remain equal by time-sharing if the following identity holds:

$$\lambda - \mu_1 \dot{T}_1 = \mu_4 - \mu_2(1 - \dot{T}_1);$$

or, equivalently, if

$$\dot{T}_1 = \frac{\lambda + \mu_2 - \mu_4}{\mu_1 + \mu_2}.$$

This identity can hold if the expression for \dot{T}_1 is in $[0, 1]$, i.e., if $\lambda + \mu_2 > \mu_4$ and $\lambda < \mu_1 + \mu_4$. The latter inequality certainly holds because of the utilization conditions.

Thus, if $\lambda + \mu_2 > \mu_4$, there is an increase move in which S_t goes from state $(1, 4)$ to the feasible state $(1, 2, 4)$. On the other hand, if $\lambda + \mu_2 < \mu_4$, state S_t jumps from state $(1, 4)$ to state $(2, 4)$, since $(1, 2, 4)$ is infeasible. Below, we will show these two transitions as

$$(1, 4) \rightarrow (1, 2, 3) \text{ and } (1, 4) \rightarrow (1, 2, 4)_0 \rightarrow (2, 4),$$

respectively, where $(1, 2, 4)_0$ indicates that $(1, 2, 4)$ is not feasible.

If S_t reaches the state in which all the queues are empty, S_t remains in the absorbing “zero” state. Note that the zero state is always feasible by the utilization conditions. In this state, $\dot{T}_i = \nu_i/\mu_i$ is acceptable by Equation (5) and leads to $\dot{X}_i = 0$, $\forall i$, $1 \leq i \leq K$.

Since the zero state is feasible, S_t remains in this state for a positive amount of time. Now since no new maximum will appear in the network in the zero state, the fluid limit equations will not change, and S_t remains in this absorbing state forever.

Lemma 3.1. *At a regular point t , $\dot{X}_i(t) = \dot{X}_j(t)$ if i and j are in the set of maxima of same group.*

Proof. Since i and j are in the same group and both in the set of maxima, $X_i = X_j$. Thus, if t is a regular point, X_i and X_j are differentiable and $\dot{X}_i(t) = \dot{X}_j(t)$.

Lemma 3.2. *Consider one arbitrary queue i and assume that t is a regular point such that $X_i(t) = 0$. Let ν_{in} be the total arrival rate and ν_{out} be the departure rate of queue i at time t . Then $\nu_{in} = \nu_{out}$.*

Proof. Since t is regular and $X_i(t) = 0$, X_i is differentiable at t and $\dot{X}_i(t) = 0$. This proves that $\nu_{in} = \nu_{out}$.

Lemma 3.3. *At a regular point, α cannot be a non-negative vector.*

Proof. Define matrix $\mathbf{E}_{L \times J}$ as the following.

$$\mathbf{E} = [\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_J] \quad (7)$$

where \mathbf{e}_j is the column vector of length L defined as follows:

$$\mathbf{e}_j = [\mathbf{0}_{\sum_{k=1}^{j-1} L_k}, \mathbf{1}_{L_j}, \mathbf{0}_{\sum_{k=j+1}^J L_k}]^T. \quad (8)$$

By the fluid model equations (1) and Lemma 3.1, we have the following matrix equation:

$$\begin{bmatrix} -(\mathbf{D}_L)^T & \mathbf{E} \\ \mathbf{E}^T & \mathbf{0}_{J \times J} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{T}} \\ \alpha \end{bmatrix} = \begin{bmatrix} \lambda_L \\ \mathbf{1}_J \end{bmatrix}. \quad (9)$$

Solving Equation (9) using the block inverse formula, we have

$$\alpha = (\mathbf{E}^T (\mathbf{D}_L)^{-T} \mathbf{E})^{-1} (\mathbf{E}^T (\mathbf{D}_L)^{-T} \lambda_L + \mathbf{1}_J). \quad (10)$$

A careful observation shows that $\mathbf{E}^T (\mathbf{D}_L)^{-T} \lambda_L + \mathbf{1}_J$ is a positive vector, by the stability condition of these subset of queues which is a weaker condition than the stability condition of the whole network.

In Section 2.2, we showed that \mathbf{D}^{-T} is a negative matrix. \mathbf{D}_L has the same properties as \mathbf{D} so, $\mathbf{E}^T(\mathbf{D}_L)^{-T}\mathbf{E}$ is also a negative matrix. Now since $\mathbf{E}^T(\mathbf{D}_L)^{-T}\mathbf{E}\boldsymbol{\alpha}$ is a positive vector, $\boldsymbol{\alpha}$ cannot be a non-negative vector.

Lemma 3.3 shows that some non-zero maxima must decrease, so that S_t cannot remain in any state but the zero state which is the only absorbing state.

A sufficient condition for reaching the zero state is that all the possible states of the network are feasible. Indeed, the drifts of the maximum queues in a group are equal in a feasible state, so that a queue cannot disappear from the set of maxima before its group becomes empty. Therefore, S_t visits a new state each time a new maximum appears, and it cannot make a loop. Consequently, after some finite time S_t reaches the zero state and stability is proved.

In some networks, some states are not feasible. In such networks, proving that S_t does not loop requires a different argument. We show it for networks with two groups of two queues.

3.2. Lu-Kumar network with two groups of two queues

In this section, we prove Theorem 2.1. The method of proof is to investigate the possible paths of S_t in the state diagram to show that S_t cannot make a loop and necessarily reaches the zero state.

The proof uses a few properties of the trajectory of S_t . We begin with the following result that says that if a queue is removed from the set of maxima, the remaining maxima in its group have a positive drift.

Lemma 3.4. *Suppose that S_t switches from state $S^{(1)}$ to $S^{(2)}$. Let $S_\Delta = S^{(1)} \setminus S^{(2)}$. Then, if $i \in S_\Delta$, the maxima of the group that i belongs to have positive drift in state $S^{(2)}$.*

Proof. If $i \in S_\Delta$, queue i is removed from the set of maxima after jumping to state $S^{(2)}$. This requires that, in state $S^{(2)}$, queue i has a smaller drift than the other maxima in its group, say \mathcal{G}_j . That is, $\dot{X}_i < \dot{X}_k, \forall k \in S^{(2)} \cap \mathcal{G}_j$. But, since queue i is not served, one has $\dot{X}_i \geq 0$. Consequently, the drift of the maxima in group j is positive.

Also recall that a queue cannot disappear from the set of a maxima by a transition

out of a feasible state.

We are now ready to prove the following result.

Lemma 3.5. *Under the stability conditions, there is no loop in the trajectories of the state diagram of the Lu-Kumar network with two groups of two queues.*

Proof. We prove the lemma by contradiction. To have a loop in the state diagram, there are three possible scenarios.

Scenario I: There is an empty group in one of the states in the loop.

Without loss of generality, we can consider this state to be $(1, \emptyset)$ or $(1, 2, \emptyset)$.

To show that this is not possible, we first show that $(1, 2, \emptyset)$ cannot be feasible. To see this, assume that $(1, 2, \emptyset)$ is feasible. Since a queue cannot disappear from the set of maxima by leaving a feasible state, the only possible transitions starting from $(1, \emptyset)$ or $(1, 2, \emptyset)$ are

$$(1, \emptyset) \rightarrow (1, 2, \emptyset) \rightarrow (\emptyset, \emptyset) \text{ or } (1, \emptyset) \rightarrow (\emptyset, \emptyset).$$

These transitions lead to the absorbing zero-state, which contradicts the existence of a loop. Hence, $(1, 2, \emptyset)$ is not feasible and $(1, \emptyset)$ must be the feasible state with an empty group in the loop.

Second, we claim that $(1, 2, 3, 4)$ cannot be feasible. Assume that it is. One then has the following transitions starting from $(1, \emptyset)$:

$$(1, \emptyset) \rightarrow (1, 2, \emptyset)_0 \rightarrow (1, 2, 3, 4).$$

In $(1, 2, 3, 4)$, the drift of group 2 is positive since it was previously empty so queues 3 and 4 are necessarily increasing. Thus, group 1 has negative drift and next becomes empty. We claim that the next feasible state is then $(\emptyset, 3, 4)$. Indeed, if that state is not feasible, the next feasible state would then be $(\emptyset, 3)$ (without loss of generality). But, in that state, the maximum 3 must decrease (since group 1 has a zero drift and one of the groups must have a negative drift, by Lemma 3.3) and the other queue 4 does not, which is not possible. Thus, the next transitions must be

$$(1, 2, 3, 4) \rightarrow (\emptyset, 3, 4) \rightarrow (\emptyset, \emptyset),$$

which contradicts the existence of a loop.

Third, we show that S_t must go to the zero state. Indeed, we have the following transitions:

$$(1, \emptyset) \rightarrow (1, 2, \emptyset)_0 \rightarrow (1, 2, 3, 4)_0.$$

The next state is either $(1, 2, 3)$ or $(1, 2, 4)$. To see this, first note that if one of queues 1 or 2 are removed from the set of maxima, by Lemma 3.4, group 1 has positive drift in the next state. On the other hand, since group 2 was previously empty, it necessarily has a positive drift in the next state. Since both of the groups cannot have a positive drift, this is a contradiction and it follows that 1 and 2 remain in the set of maxima, and group 1 has negative drift. Therefore, without loss of generality we can consider the next transition to be to $(1, 2, 3)$, i.e., we have the following transitions:

$$(1, \emptyset) \rightarrow (1, 2, \emptyset)_0 \rightarrow (1, 2, 3, 4)_0 \rightarrow (1, 2, 3).$$

Next, group 1 becomes empty and S_t visits $(\emptyset, 3)$. After some time, queues 3 and 4 become equal, as 3 has negative drift, so S_t visits $(\emptyset, 3, 4)$, so that group 2 has positive drift by Lemma 3.4. We claim that $(\emptyset, 3, 4)$ must be feasible. Indeed, group 2 has a positive drift, so group 1 cannot have a positive drift and queues 1 and 2 cannot appear in a new set of maxima by Lemma 3.4. Since $(\emptyset, 3, 4)$ is feasible, the next state is the zero state, which contradicts the existence of a loop.

Scenario II: None of the states in the loop has an empty group and there is a state of cardinality 3 in the loop. Note that state $(1, 2, 3, 4)$ cannot be in the loop since from that state one of the groups becomes empty. In order to have a loop containing a state with cardinality 3, we necessarily have a jump happening from a state with 3 maxima, since state $(1, 2, 3, 4)$ cannot be in the loop. Without loss of generality suppose that this state is $S^{(1)} = (1, 2, 3)$ and queue 4 is the new maximum that appears.

Case 1: The drift of queue 3 at $S^{(1)}$ is positive, so necessarily drifts of 1 and 2 are equal and negative by Lemma 3.3. Therefore, we can only jump to $S^{(2)} = (1, 2, 4)$ with drift of 4 positive and drifts of 1 and 2 negative by Lemma 3.4. From this point, no new maximum can appear until group 1 goes empty since $\dot{X}_4 > \dot{X}_3 > 0$ and we are back to Scenario I.

Case 2: The drift of queue 3 at $S^{(1)}$ is negative. To have a jump from this state, S_t can either go to $(1, 2, 4)$ with $\dot{X}_4 > 0$ in which we are in the same situation as Case 1.



FIGURE 3: Loop in state diagram of Lu-Kumar network

The other possibility is that $\dot{X}_3 = \dot{X}_4 < 0$, and one of the queues 1 or 2 (for example queue 2) is removed from the set of maxima. Note that by Lemma 3.4, queues 3 and 4 remain equal while having negative drift. From this point, no new maximum can appear until group 2 goes empty since $\dot{X}_1 > \dot{X}_2 > 0$ and we are back to Scenario I.

Scenario III: All of the states in the loop are of cardinality 2. Obviously, not all the maxima of a state can change in a jump. So for example from $S = (1, 3)$, there is no jump to $S = (2, 4)$. Now we consider 3 cases:

Case 1: There is a loop of length 2. Without loss of generality we can consider the loop to be $(1, 3) \rightarrow (1, 4) \rightarrow (1, 3)$. But at state $(1, 4)$, it is not possible that queue 3 appears again as the new maximum since $\dot{X}_4 > \dot{X}_3 > 0$. So there is no loop of length 2 in the state diagram.

Case 2: There is a loop of length 3. Without loss of generality we can consider the loop to be $(1, 3) \rightarrow (1, 4) \rightarrow (2, 4) \rightarrow (1, 3)$. But clearly the last jump in the loop is not valid. So there is no loop of length 3.

Case 3: There is a loop of length 4. Without loss of generality we can consider the loop to be $(1, 3) \rightarrow (1, 4) \rightarrow (2, 4) \rightarrow (2, 3) \rightarrow (1, 3)$. We cannot rule out this case just by inspection like the previous cases. Note that in each of these states the drift of one group is positive and the other one negative since there cannot be a jump to a state where both of the drifts are negative by Lemma 3.4. A careful observation shows that the sign of the drifts of each group should change in every jump. If not, as we can see in Figure 3, the last jump $(2, 3) \rightarrow (1, 3)$ is not valid by Lemma 3.4.

The last possible case that we have to investigate is the loop with the drifts shown in Figure 4.

The possibility of this loop happening cannot be rejected just by inspection. We write the exact conditions leading to each jump which is an inequality, then sum them

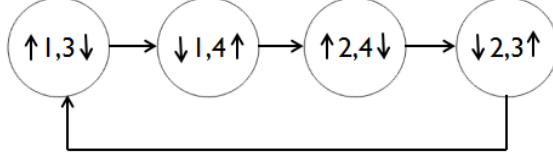


FIGURE 4: Loop in the state diagram of Lu-Kumar network

up and reach a contradiction. The jump $(\uparrow 1, 3 \downarrow) \rightarrow (\downarrow 1, 4 \uparrow)$ implies that state $S = (1, 3, 4)$ is not feasible. Indeed solving equation $\dot{X}_3 = \dot{X}_4$, where

$$\begin{aligned}\dot{X}_3 &= \lambda_3 - \mu_3 \dot{T}_3 + \mu_1 r_{13} + \mu_3 \dot{T}_3 r_{33} + \mu_4 (1 - \dot{T}_3) r_{43} \\ \dot{X}_4 &= \lambda_4 - \mu_4 (1 - \dot{T}_3) + \mu_1 r_{14} + \mu_3 \dot{T}_3 r_{34} + \mu_4 (1 - \dot{T}_3) r_{44},\end{aligned}$$

leads to $\dot{T}_3 < 0$ which is not acceptable. So for $\dot{T}_3 = 0$, $\dot{X}_3 < \dot{X}_4$ and the following condition is obtained.

$$\lambda_4 - \mu_4 + \mu_1 r_{14} - \lambda_3 - \mu_1 r_{13} - \mu_4 r_{43} + \mu_4 r_{44} > 0 \quad (11)$$

Similarly, for the other 3 jumps we obtain 3 inequalities which are the following.

$$\lambda_2 - \mu_2 + \mu_4 r_{42} - \lambda_1 - \mu_4 r_{41} - \mu_2 r_{21} + \mu_2 r_{22} > 0 \quad (12)$$

$$\lambda_3 - \mu_3 + \mu_2 r_{23} - \lambda_4 - \mu_2 r_{24} - \mu_3 r_{34} + \mu_3 r_{33} > 0 \quad (13)$$

$$\lambda_1 - \mu_1 + \mu_3 r_{13} - \lambda_2 - \mu_3 r_{32} - \mu_1 r_{12} + \mu_1 r_{11} > 0 \quad (14)$$

Adding inequalities (11) to (14), we have

$$\begin{aligned}& -\mu_1(1 - r_{11} + r_{12} + r_{13} - r_{14}) - \mu_2(1 + r_{21} - r_{22} + r_{23} - r_{24}) \\ & -\mu_3(1 - r_{31} + r_{32} - r_{33} + r_{34}) - \mu_4(1 + r_{41} - r_{42} + r_{43} - r_{44}) > 0,\end{aligned}$$

which is clearly not true so we reach a contradiction and the proof of the lemma is complete.

The immediate result of Lemma 3.5 is that S_t will reach the zero state after a finite time since S_t cannot travel around a finite state diagram without making a loop and reaching the zero state.

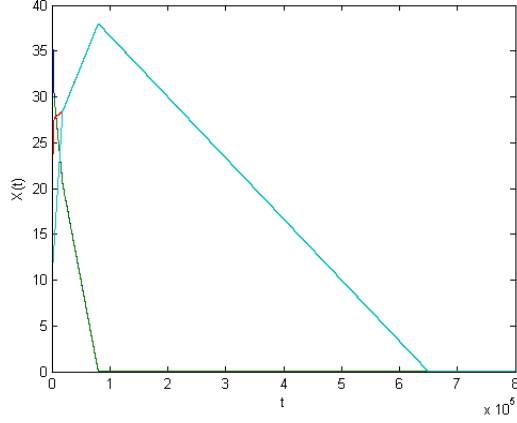


FIGURE 5: Trajectories of fluid in basic Lu-Kumar network

We show the simulation results for the Lu-Kumar network of Figure 1 in the fluid limit. In the simulation, service rates are $\boldsymbol{\mu} = [3, 1, 1, 1]^T$, and arrival rate is $\lambda = 0.4$. Initial queue-lengths is $\mathbf{X}(0) = [40, 30, 20, 10]^T$. Figure 5 shows the trajectories of the fluid in different queues versus time. As we can see, first the queue-lengths in different groups become equal. Then group 1 goes empty, and from that point drift of group 2 is negative until it is empty.

4. LDQ Scheduling

4.1. Policy

In each group, we define a queue to be “dominating” if it cannot feed a longest queue in a different group that is larger than itself. Recall that S is the set of maxima in the network. For each group j , the dominating set \mathcal{D}_j is defined to be

$$\mathcal{D}_j = \{i \in \mathcal{G}_j : r_{is} = 0 \text{ if } X_i < X_s, \forall s \in S\}$$

The scheduling policy is to serve the longest queue in \mathcal{D}_j . If $\mathcal{D}_j = \emptyset$, do not serve any queues from group j . As an example, see Figure 6, where queue 3 is the maximum with length 30. Since jobs leaving both queues 1 and 2 can be destined to queue 3, $\mathcal{D}_1 = \emptyset$ so no queues in station 1 is served. As we can see, the policy is not work-conserving. The policy is robust to the knowledge of service rates and exact values of routing probabilities. The draw-back in comparison with LQ scheduling is that it

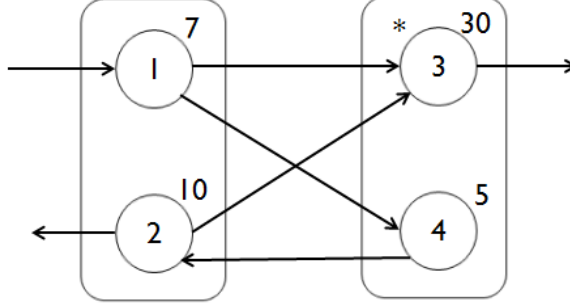


FIGURE 6: LDQ is not work-conserving

needs global knowledge of queue lengths and the topology of the network.

4.2. Stability proof

The fluid model equations for this policy is the same as Equation 1. The scheduling policy requires that

$$\sum_{i \in S_j} \dot{T}_i(t) \leq 1 \quad (15)$$

$$\sum_{i \in \mathcal{G}_j \setminus S_j} \dot{T}_i(t) = 0. \quad (16)$$

Note that since the policy is not work-conserving, we have inequality in Equation (15) instead of equality for LQ scheduling.

We prove the stability using the Lyapunov function $V(\mathbf{X}) = \max\{X_i\}$. By the policy, no jobs will be scheduled destined to the queue with maximum length in the whole network. (not necessarily the maximum in one group) So, if the maximum is unique $\dot{V}(\mathbf{X})$ is clearly negative. We prove that if the maximum is not unique, in a regular point the drift of the set of maxima are equal and negative. Let S' be the set of maximum-length queues in the network, and $|S'| = L'$. A regular point is a value of $t \in [0, \infty)$ at which the function

$$f : [0, \infty) \rightarrow \mathbb{R}, \quad f(t) = \max_i \{X_i\}_{1 \leq i \leq K}$$

is differentiable. Suppose that $L' > 1$. In a regular point, by Lemma 3.1, the drifts of all the queues in S' are equal. Since there is no flow to the sub-network of queues in S'

coming from other queues, we can only consider this sub-network to analyse the drift of the Lyapunov function. Let the corresponding drift matrix and arrival vectors to set S' be $\mathbf{D}_{L'}$ and $\boldsymbol{\lambda}_{L'}$. Suppose that J' groups ($J' \leq J$) have queues in S' so $S' = \cup_{j=1}^{J'} S'_j$. Let $L'_j = |S'_j|$. In a regular point, the drift of the queues in this sub-network S' are all equal to α .

Lemma 4.1. $\dot{f}(t) = \alpha < 0$.

Proof. Define matrix $\mathbf{E}_{L' \times J'}$ as the following.

$$\mathbf{E} = [\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{J'}], \quad (17)$$

where \mathbf{e}_j is a column vector of length L' .

$$\mathbf{e}_j = [\mathbf{0}_{\sum_{k=1}^{j-1} L'_k}, \mathbf{1}_{L'_j}, \mathbf{0}_{\sum_{k=j+1}^{J'} L'_k}]^T \quad (18)$$

The matrix equation will be

$$\begin{bmatrix} -\mathbf{D}'_L & \mathbf{E} \\ \mathbf{E}^T & \mathbf{0}_{J' \times J'} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{T}}_{L'} \\ \alpha \mathbf{1}_{J'} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\lambda}_{L'} \\ \mathbf{t}_{J'} \end{bmatrix} \quad (19)$$

where vector $\mathbf{t}_{J'} = [t_j]$, $t_j = \sum_{i \in S'_j} \dot{T}_i \leq 1$. Note that for a work-conserving policy $\mathbf{t}_J = \mathbf{1}_J$. A careful observation shows that at least one of the elements of \mathbf{t}_J is equal to 1, if the network topology is acyclic. The reason is that the sub-network certainly has a flow that is leaving the sub-network via an “output” queue. Note that if there is a cycle in the network, such a queue may not exist. (See Figure 7) The group which contains the “output” queue is work-conserving.

Solving Equation (19) using block inverse formula, we have

$$\alpha \mathbf{1}_{J'} = (\mathbf{E}^T \mathbf{D}_{L'}^{-T} \mathbf{E})^{-1} (\mathbf{E}^T \mathbf{D}_{L'}^{-T} \boldsymbol{\lambda}_{L'} + \mathbf{t}_{J'}). \quad (20)$$

The vector $(\mathbf{E}^T \mathbf{D}_{L'}^{-T} \boldsymbol{\lambda}_{L'} + \mathbf{t}_{J'})$ is positive in at least one element corresponding to the work-conserving group by the stability condition of this sub-network. (which is a weaker condition than the stability condition of the whole network) We have already proved that $\mathbf{E}^T \mathbf{D}_{L'}^{-T} \mathbf{E}$ is a negative matrix. Now since $\mathbf{E}^T \mathbf{D}_{L'}^{-T} \mathbf{E}(\alpha \mathbf{1}_{J'})$ is not a negative vector, α cannot be non-negative.

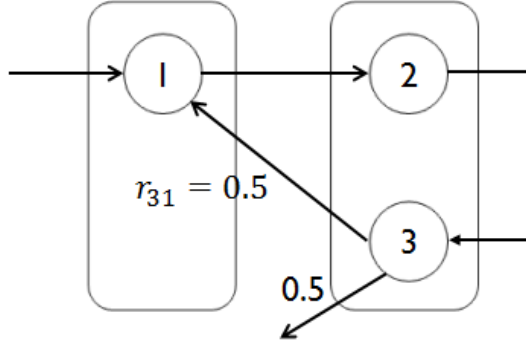


FIGURE 7: No output queue in a cycle

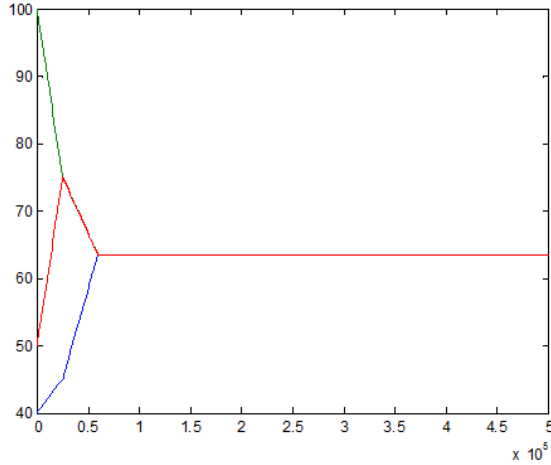


FIGURE 8: LDQ in a cycle

Consequently, $V(\mathbf{X})$ is a Lyapunov function and proof of Theorem 2.2 is complete.

The simulation results show that the network shown in Figure 7 is indeed unstable under LDQ scheduling. In this simulation all the service rates are 1, and the arrival rate is 0.2. The result is shown in Figure 8. As one can see, first all the queues become equal but then do not go to 0 and remain constant in this example.

The basic intuition behind this fact is the following. Consider the simple case of Figure 9. We have one queue in each station so no scheduling is needed. However, LDQ serves only one of the queues at a time, and they are virtually in the same group. Consequently, LDQ is not throughput optimal if we have a cycle in the network.

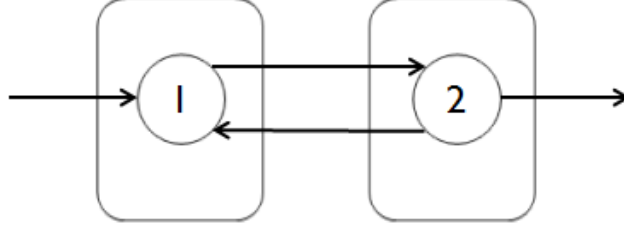


FIGURE 9: A simple cycle

In the end, we see some simulation results for the basic Lu-Kumar network under LDQ scheduling. The network topology is acyclic so we expect LDQ to be stable. Simulation results shown in Figure 10 verifies this. In this simulation, all the service rate are 1, and the arrival rate is 0.4. As one can see, first all the queues become equal and then go to 0 together.

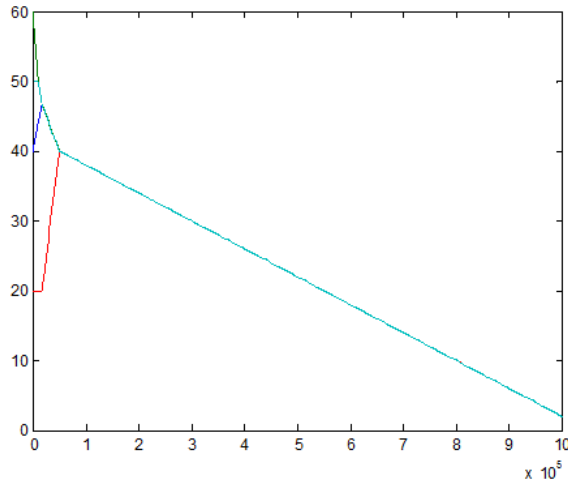


FIGURE 10: LDQ in basic Lu-Kumar example

Acknowledgements

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References

- [1] LU, S. H., KUMAR, P. R. (1991). Distributed scheduling based on due dates and buffer priorities. *IEEE Trans. Automatic Control* **36**, 1406–1416.
- [2] TASSIULAS, L., EPHERMIDES, A. (1992). Dynamic server allocation to parallel queues with randomly varying connectivity. *Technical Research Report* ISR-TR 298, University of Maryland.
- [3] TASSIULAS, L., EPHERMIDES, A. (1992). Stability properties of constrained queueing systems and scheduling policies for maximum throughput in multihop radio networks. *IEEE Trans. Automatic Control* **37**, 1936–1948.
- [4] MALYSHEV, V. A. (1993). Networks and dynamical systems. *Adv. Appl. Prob.* **25**, 140–175.
- [5] KUMAR, P. R., MEYN, S. P. (1995). Stability of queueing networks and scheduling policies. *IEEE Trans. Automatic Control* **40**, 251–260.
- [6] DAI, J. G. (1995). On positive Harris recurrence of multiclass queueing networks: a unified approach via fluid limit models. *Ann. Appl. Prob.* **5**, 49–77.
- [7] MCKEOWN, N. (1995). Scheduling algorithms for input-queued cell switches. Doctoral Thesis, University of California, Berkeley.
- [8] STOYLAR, A. L. (1995). On the stability of multiclass queueing networks: a relaxed condition via limiting fluid processes. *Markov Process. Relat. Fields* **1**, 491–512.
- [9] DAI, J. G. (1999). *Stability of fluid and stochastic processing networks*, MaPhySto Miscellanea Publication, No. 9.
- [10] KUMAR, S., GIACCONE, P., LEONARDI E. (2002). Rate stability of stable marriage scheduling algorithms in input-queued switches. *Proc. 40th Annual Allerton Conference on Computers, Communications, and Control*. University of Illinois, Urbana-Champaign, IL.
- [11] DIMAKIS, A., WALRAND, J. (2006). Sufficient conditions for stability of longest-queue-first scheduling: second-order properties using fluid limits. *Adv. Appl. Prob.* **38**, 505–521.
- [12] BRAMSON, M. (2008). Stability of queueing networks. *Probability Surveys* **5**, 169–345.
- [13] LIN, X., SHROFF, N. (2009). Understanding the capacity region of the greedy maximal scheduling algorithm in multi-hop wireless networks. *IEEE/ACM Trans. on Networking* **17**, 1132–1145.
- [14] BAHARIAN, G., TEZCAN, T. (2011). Stability analysis of parallel server systems under longest queue first. *Math. Meth. Oper. Res.* **74**, 257–279.